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NONPARAMETRIC ESTIMATION OF THE PROBABILITY OF A LONG DELAY IN THE M/G/1 QUEUE

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An M/G/1 queue is approached by stationary Poisson traffic with known arrival rate. Observations of service times are all that is known about the service distribution. Nonparametric estimates of the probability of a long customer delay are given. The estimates include the solution of an equation involving the empirical transform of the service times. Asymptotic properties of the estimates are derived. Simulation studies of the small sample behavior of the estimates are reported. The jackknife is used to provide error assessment of the estimates and to construct confidence intervals in the simulation studies of small sample behavior.

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NONPARAMETRIC ESTIMATION OF THE PROBABILITY OF
A LONG DELAY IN THE M/G/1 QUEUE

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1. Introduction

The application of probability theory to a wide variety of congestion problems has been well catalogued in many papers. Most of the elegant solutions obtained are in somewhat implicit form, being presented as functional equations, or, frequently, as integral (Laplace) transforms, generating functions, and sometimes as combinations of the above. The results obtained naturally appear in terms of component distribution functions and stochastic processes (renewal, Poisson etc). Only rarely are issues addressed that arise when actual data is to be used as a basis for inference from the models; however, see Cox [1965].

In this paper we consider the nonparametric estimation of the probability of a long customer delay in an M/G/1 system, given a known Poisson arrival rate λ and observations of independent service times from the service distribution, presumed unknown. Although the approach and results are given concretely for the M/G/1 system, they apply more widely.

To be specific, consider a single server system approached by a stationary Poisson (λ) traffic with λ known. Service times, X are independent identically distributed with unknown distribution function F_X; assume $\lambda E[X] = \rho < 1$. Let observations of the service times be all that is known about F; denote these by $x_1, x_2, \ldots, x_n$. The objective is to supply nonparametric estimates, and error assessments thereof, of the probability of a long delay experienced by an arriving customer.

It is well known that if W(t) is the virtual waiting time in the M/G/1 queue and $\rho < 1$, then the moment generating function

$$
\begin{aligned}
E[e^{sW}] &= \lim_{t\to\infty} E[e^{sW(t)}] \qquad (1.1)\\
&= (1-\rho)\left\{1 - \rho\left[\frac{E[e^{sX}] - 1}{sE[X]}\right]\right\}^{-1}\\
&= (1-\rho)[1 - \rho A(s)]^{-1}
\end{aligned}
$$

where A(s) is the moment generating function of a distribution H. If A(s) exists for $s < s_0$, $s_0 > 0$, then there will be a smallest real zero $s = \kappa > 0$ of the denominator of (1.1) which can be used to show that

$$
P\{W > w\} \sim D(\kappa)\, e^{-\kappa w}, \qquad w \to \infty. \qquad (1.2)
$$

We will always assume this is the case.

One way of establishing (1.2) is to introduce

$$
\Psi(s) = \frac{E[e^{sW} - 1]}{s} = \int_0^\infty P\{W > w\}\, e^{sw} dw \qquad (1.3)
$$

into (1.1) and to rewrite in the form

$$
\Psi(s) = \rho\frac{[A(s) - 1]}{s} + \rho A(s)\Psi(s) \qquad (1.4)
$$

which is equivalent to the terminating renewal equation, see Feller [1966] p. 362,

$$\bar{F}_W(w) = P\{W > w\} = \rho\bar{H}(w) + \rho \int_0^w P\{W > w-x\} \, H(dx) \tag{1.5}$$

where

$$H(w) = \int_0^w \frac{P\{X > y\}}{E[X]} \, dy. \tag{1.6}$$

Following Feller, introduce $\bar{F}_W^{\#}(w) = \bar{F}_W(w) \, e^{\Theta w}$, Θ real and positive, into (1.5) to obtain

$$\bar{F}_W^{\#}(w) = \rho \, \bar{H}^{\#}(w) + \int_0^w \bar{F}^{\#}(w-x) \, \rho e^{\theta x} H(dx). \tag{1.7}$$

Choose $\theta = \kappa$ so that

$$\int_0^\infty \rho e^{\theta x} H(dx) = \int_0^\infty H^{\#}(dx) = 1 \tag{1.8}$$

yielding a standard renewal equation for $\bar{F}_W^{\#}$. From the key renewal theorem, it follows that

$$\lim_{w\to\infty} \bar{F}_W^{\#}(w) = \lim_{w\to\infty} \bar{F}_W(w)e^{\kappa w} = \frac{c(\kappa)}{m(\kappa)} \tag{1.9}$$

where

$$c(\kappa) = \rho \int_0^\infty e^{\kappa x} \bar{H}(x)dx \tag{1.10}$$

and

$$m(\kappa) = \rho \int_0^\infty xe^{\kappa x} H(dx). \tag{1.11}$$

Summarizing

$$P\{W > t\} \sim \frac{c(\kappa)}{m(\kappa)} e^{-\kappa t} \qquad t \to \infty \tag{1.12}$$

where κ is the positive solution to the equation

$$\lambda\theta^{-1}[\phi(\theta) - 1] = 1 \tag{1.13}$$

with

$$\phi(\theta) = E[e^{\theta X}]. \tag{1.14}$$

In section 2, a nonparametric estimate, $\hat{\kappa}$, of κ is studied which solves equation (1.13) with the empirical moment generating function replacing ϕ. This estimate is related to an estimate studied by Stigler [1971] in the context of estimating the probability of extinction for branching processes. $\hat{\kappa}$ is shown to be a consistent estimate of κ having a distribution in the domain of attraction of a stable law. Under certain conditions, a central limit theorem for $\hat{\kappa}$ can be obtained.

In section 3 a nonparametric estimate of

$$p(t) \cong \frac{c(\kappa)}{m(\kappa)} e^{-\kappa t} \tag{1.15}$$

is given.

In Section 4, some results of simulation studies of the estimates of κ and $p(t)$ are presented.

2. Nonparametric Estimation of the Exponent κ of the Probability of a Long Wait

Assume for the remainder of the paper that $\lambda=1$ is known. Let $x_1, \ldots, x_n$ be the observations of the service times. A non-parametric estimate of the moment generating function of the service time distribution is

$$\hat{\phi}(\theta) = \frac{1}{n} \sum_{i=1}^{n} e^{\theta x_i}. \tag{2.1}$$

The sample equivalent to equation (1.13) is thus

$$1 = \theta^{-1}[\hat{\phi}(\theta)-1]. \tag{2.2}$$

At $\theta = 0$, the RHS of (2.2) is $\bar{x}$ which is less than 1 if $\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} < 1$; the data can only be analyzed for a stationary model under this assumption, which will be made in what follows. Further, (2.2) has a unique solution, $\hat{\kappa}$, which is an estimate of κ. A three-term Taylor's expansion of the RHS of (2.2) about $\theta=0$ gives an initial estimate

$$\hat{\kappa}_H = 2[1 - \bar{x}]\,(\overline{x^2})^{-1} \qquad (2.3)$$

where $\overline{x^k} = \frac{1}{n}[x_1^k + \ldots + x_n^k]$, the k^{th} sample moment. Since $\theta^{-1}[\hat{\phi}(\theta) - 1] \geq \bar{x} + \frac{1}{2}\theta\,\overline{x^2}$, $\hat{\kappa}_H$ is an upper bound for $\hat{\kappa}$. Equation (2.2) can be solved via search or Newton-Raphson iteration to obtain the estimate $\hat{\kappa}$.

We will now present asymptotic results for the distribution of $\hat{\kappa}$ as the sample size $n \to \infty$. By assumption, if X is a random variable having the service time distribution, then $E[e^{B'X}] < \infty$ for some $B' > \kappa > 0$. Thus, there is $B > \kappa$ such that for all $0 < b \leq B$

$$E[X^n e^{bX}] < \infty. \qquad (2.4)$$

It follows from the monotonicity of the RHS of (2.2) that

$$\begin{aligned} P\{\hat{\kappa} > B\} &= P\left\{\frac{\hat{\phi}(B) - 1}{B} - \frac{[\phi(B) - 1]}{B} < 1 - \frac{[\phi(B) - 1]}{B}\right\} \\ &= P\left\{\hat{\phi}(B) - \phi(B) < B[1 - \frac{[\phi(B) - 1]}{B}]\right\}. \end{aligned} \qquad (2.5)$$

Since $\phi(\theta)$ is a monotone function and $B > \kappa$, $[1 - B^{-1}[\phi(B) - 1]]$ is negative. Thus, by the strong law of large numbers

$$\lim_{n\to\infty} P\{\hat{\kappa} > B\} = 0. \qquad (2.6)$$

Let

$$\hat{f}(\theta) = 1 - \theta^{-1}[\hat{\phi}(\theta) - 1]. \qquad (2.7)$$

Expand $\hat{f}(\hat{\kappa})$ in Taylor's series about the solution κ of (1.13). Since $\hat{f}(\hat{\kappa}) = 0$,

$$0 = \hat{f}(\kappa) + (\hat{\kappa}-\kappa)\ \hat{f}'(\kappa) + \frac{1}{2}\ (\hat{\kappa}-\kappa)^2\ \hat{f}''(\theta\kappa) \qquad (2.8)$$

for some θ. Thus

$$\hat{\kappa} - \kappa = -\hat{f}(\kappa)[\hat{f}'(\kappa) + \frac{1}{2}\ (\hat{\kappa}-\kappa)\ \hat{f}''(\theta\kappa)]^{-1}; \qquad (2.9)$$

$$E[\hat{f}(\kappa)] = 0;$$

$$E[\hat{f}'(\kappa)] = \frac{1}{\kappa^2}\ E[-\kappa X e^{\kappa X} + e^{\kappa X} - 1] = \beta < 0. \qquad (2.10)$$

By the strong law of large numbers

$$\lim_{n\to\infty} \hat{f}(\kappa) = 0 \qquad (2.11)$$

and

$$\lim_{n\to\infty} \hat{f}'(\kappa) = \beta < 0 \qquad (2.12)$$

with probability 1. It follows from (2.6), (2.9), (2.11) and (2.12) that $\hat{\kappa}$ converges in probability to κ as $n \to \infty$. Thus, $\hat{\kappa}$ is a consistent estimate of κ.

If $E[e^{2\kappa X}] < \infty$, then

$$\mathrm{Var}[\hat{f}(\kappa)] = \frac{1}{n}\frac{1}{\kappa^2}\mathrm{Var}[e^{\kappa X}] \equiv \frac{1}{n} r^2. \tag{2.13}$$

By the central limit theorem

$$\frac{\beta}{r}\sqrt{n}\,\{\hat{\kappa} - \kappa\} = \frac{\beta}{r}\sqrt{n}\,[-\hat{f}(\kappa)]\,[\hat{f}'(\kappa) + \frac{1}{2}(\hat{\kappa}-\kappa)\,\hat{f}''(\theta\kappa)]^{-1} \tag{2.14}$$

is asymptotically standard normal as $n \to \infty$.

If $E[e^{2\kappa X}] = \infty$, then the distribution G of $e^{\kappa X}$ is in the domain of attraction of a stable law with index $1 < \alpha < 2$, see Feller [1966]. Let $\{a_n\}$ be a sequence of numbers such that

$$\frac{n}{a_n^2}\int_0^{a_n} x^2\, G(dx) \to 1 \qquad \text{as } n \to \infty. \tag{2.15}$$

The normalized random variable

$$\frac{n}{a_n}[\hat{\kappa} - \kappa] = \frac{n}{a_n}[-\frac{1}{n}\sum_{i=1}^{n} e^{\kappa X_i}][\hat{f}'(\kappa) + \frac{1}{2}(\hat{\kappa} - \kappa)\,\hat{f}''(\theta\kappa)]^{-1} \tag{2.16}$$

is in the domain of a stable law with index α, where $1 < \alpha < 2$ is such that $\int_0^y x^2 G(dx) \sim y^{2-\alpha} L(y)$ where L is a slowly varying function.

To summarize, we have the following result.

PROPOSITION

a. $\hat{\kappa}$ is a consistent estimate of κ.

b. If $E[e^{2\kappa X}] < \infty$, then the distribution of $\hat{\kappa}-\kappa$ is asymptotically normal.

c. If $E[e^{2\kappa X}] = \infty$, then $\hat{\kappa}-\kappa$ is in the domain of attraction of a stable law.

We will suppose for the remainder of this section that the service time distribution is exponential with mean $\frac{1}{\mu} > \frac{1}{2}$. In this case, (since $\lambda=1$ by assumption),

$$\kappa = \mu - 1; \tag{2.17}$$

$$\mathrm{Var}[e^{\kappa X}] = \mu(\mu-1)^2 (2-\mu)^{-1}; \tag{2.18}$$

and

$$E[\hat{f}'(\kappa)] = -1. \tag{2.19}$$

From the central limit theorem (2.14)

$$n\mathrm{Var}(\hat{\kappa}) \sim \left(\frac{r}{\beta}\right)^2 = \mu(2-\mu)^{-1} = \frac{\kappa+1}{1-\kappa}. \tag{2.20}$$

Therefore, if $g(x) = (1-x^2)^{1/2} - 1 - \sin^{-1}(-x)$, then

$$nVar[g(\hat{\kappa})] \sim [g'(\kappa)]^2 nVar[\hat{\kappa}] \sim 1.$$

Thus, g is an asymptotically variance-stabilizing transform of $\hat{\kappa}$. The simpler related transformation $\ln(1+\hat{\kappa})$ is used in the simulations of $\hat{\kappa}$ reported in Section 4.

3. A Nonparametric Estimate of P{W > t}

The asymptotic analytic result is

$$P\{W > t\} \sim \frac{c(\kappa)}{m(\kappa)} e^{-\kappa t} \equiv p(t) \tag{3.1}$$

as $t \to \infty$

where

$$c(\kappa) = \int_0^\infty e^{\kappa y} \int_y^\infty P\{X > x\}dx\, dy \tag{3.2}$$

$$= \int_0^\infty \frac{x^2}{2} P\{X \varepsilon dx\} + \frac{1}{\kappa^2} \int_0^\infty [e^{\kappa x} - 1 - \kappa x - \frac{(\kappa x)^2}{2}] P\{X \varepsilon dx\}$$

and

$$m(\kappa) = \int_0^\infty x e^{\kappa x} P\{X > x\}\, dx \tag{3.3}$$

$$= \frac{1}{\kappa^2} \left\{ \int_0^\infty \kappa x e^{\kappa x} P\{X \varepsilon dx\} - \int_0^\infty (e^{\kappa x} - 1) P\{X \varepsilon dx\} \right\}$$

$$= \frac{1}{2} \int_0^\infty x^2 P\{X \varepsilon dx\} + \frac{1}{2} \kappa \int_0^\infty x^3 P\{X \varepsilon dx\}$$

$$+ \int_0^\infty (\kappa x - 1) R(x) P\{X \varepsilon dx\}$$

where

$$R(x) = \left[e^{\kappa x} - 1 - \kappa x - \frac{(\kappa x)^2}{2}\right]. \tag{3.4}$$

An estimate of $P\{W > t\}$ is

$$\hat{p}(t) = \frac{\hat{c}(\hat{\kappa})}{\hat{m}(\hat{\kappa})} e^{-\hat{\kappa} t} \tag{3.5}$$

where $\hat{\kappa}$ is the positive solution to equation (2.2);

$$\hat{c}(\hat{\kappa}) = \frac{1}{2} \overline{x^2} + \frac{1}{\hat{\kappa}^2} \hat{R}_1(\hat{\kappa}); \tag{3.6}$$

$$\hat{m}(\hat{\kappa}) = \frac{1}{2} \overline{x^2} + \frac{1}{2} \hat{\kappa} \overline{x^3} + \hat{R}_2(\kappa); \tag{3.7}$$

$$\hat{R}_1(\hat{\kappa}) = \frac{1}{n} \sum_{i=1}^{n} \left(e^{\hat{\kappa} x_i} - 1 - \hat{\kappa} x_i - \frac{(\hat{\kappa} x_i)^2}{2}\right); \tag{3.8}$$

$$\hat{R}_2(\hat{\kappa}) = (-\hat{R}_1(\hat{\kappa})) + \hat{\kappa}\frac{1}{n}\sum_{i=1}^{n} x_i\left(e^{\hat{\kappa}x_i} - 1 - \hat{\kappa}x_i - \frac{(\hat{\kappa}x_i)^2}{2}\right). \tag{3.9}$$

The forms of $\hat{c}(\hat{\kappa})$ and $\hat{m}(\hat{\kappa})$ are chosen for the numerical stability of the ratio $\hat{c}(\hat{\kappa})\ \hat{m}(\hat{\kappa})^{-1}$.

In the remainder of this section we will assume the service time distribution is exponential with mean $\frac{1}{\mu} > \frac{1}{2}$ and $\bar{x} < 1$. In this M/M/1 queue case, it is well known that

$$p(t) = \frac{1}{\mu}\exp\{-(\mu-1)t\} = (1+\kappa)^{-1}\ e^{-\kappa t} \tag{3.10}$$

from (2.16).

We will now motivate a transformation of $\hat{p}(t)$ which is used in the simulation studies. Let $\gamma = \ln(1+\kappa)$, then $\kappa = e^{\gamma} - 1$ and

$$p(t) = \exp\{-\gamma - [e^{\gamma} - 1]t\}. \tag{3.11}$$

Let

$$h(\gamma) = \ln p(t) = -\gamma - [e^{\gamma} - 1]t \tag{3.12}$$

and $\hat{\gamma} = \ln(1+\hat{\kappa})$. A Taylor's expansion yields

$$\begin{aligned} n\mathrm{Var}[\hat{\gamma}] &= n\mathrm{Var}[\ln(1+\hat{\kappa})] \\ &\approx n\frac{1}{(1+\kappa)^2}\mathrm{Var}[\hat{\kappa}] \\ &\approx \frac{1}{1-\kappa^2} = [1-(e^{\gamma}-1)^2]^{-1} \end{aligned} \tag{3.13}$$

from (2.20). Hence,

$$\begin{aligned} n\mathrm{Var}[h(\hat{\gamma})] &\approx [-1 - e^{\gamma}t]^2 \, [1 - (e^{\gamma} - 1)^2]^{-1} \\ &= [-\,1 - t - (e^{\gamma} - 1)t]^2[1 - (e^{\gamma} - 1)^2]^{-1}. \end{aligned} \tag{3.14}$$

It follows from the definition of $h(\gamma)$ that

$$n\mathrm{Var}[h(\hat{\gamma})] \approx [-(1+t) + h(\gamma) + \gamma]^2\{1 - (e^{\gamma} - 1)^2\}^{-1} . \tag{3.15}$$

Since $0 < \kappa = e^{\gamma} - 1 < 1$, $0 < \gamma < \ln 2$. Thus, for t large

$$n\mathrm{Var}[h(\hat{\gamma})] \approx [-h(\gamma) + t]^2 \tag{3.16}$$

which suggests that the transformation $\ln[t - \ln p(t)]$ will tend to stabilize the variance of $\hat{p}(t)$, at least for exponential service time distributions.

4. Simulation Results

In this section results are presented of a simulation experiment to study small sample behavior of the estimates of κ and $P\{W > t\}$. All simulations were carried out on an IBM 3033AP computer at the Naval Postgraduate School using the LLRANDOM II random number generating package; Lewis and Uribe [1981]. In each replication 50 exponential service times are generated. If the mean of the service times is larger than 1, equation (2.2) has no positive solution. In this case, another 50 service times are generated. The estimates, $\hat{\kappa}$ of (2.2) and $\hat{p}(t)$ of equation (3.5) are computed for each replication.

Standard errors of the estimates are estimated in two ways. One is the appropriate asymptotic variance expression. The other is the jackknife procedure.

The jackknife is a procedure originally introduced by Quenouille for bias reduction [1956], and adapted by Tukey [1958] to obtain approximate confidence intervals. Suppose interest is on a parameter θ (e.g. κ or $p(t)$) that is estimated by $\hat{\theta}$ using a complex calculation from data $x_1,\ldots,x_n$. The idea is that of assessing variability by recomputing $\hat{\theta}$ after removing independent subgroups of data of equal size, and then using the recomputed $\hat{\theta}$ values to estimate a variance which is in turn applied to state a standard error or a two-sided confidence interval that contains the true θ with specified confidence. A few more details follow; for more, see Efron [1982] and his more recent work, or Mosteller and Tukey [1977]. The actual calculation involves splitting the n data points into g disjoint groups of size m; n=mg. In our simulations g is always 10. Then calculate $\hat{\theta}_{(-j)}$, $j=1,2,\ldots,g$: the estimate of θ based on a reduced data set that omits the j^{th} group. In the simulations, the first group is the first five (unordered) service times; the second group is the second five service times, etc.

Now Tukey computes pseudo-values

$$y_j = g\hat{\theta} - (g-1)\,\hat{\theta}_{(-j)}$$

which are treated as independent. Tukey recommends referring the mean of the pseudo-values $\bar{y}$ to Student's t with g-1 degrees of freedom to obtain confidence limits.

In the jackknife procedure to estimate κ, it is sometimes the case that a positive solution of equation (2.2) does not exist for the data set that omits the j^{th} subgroup because the mean of this reduced data set is larger than 1. In this case, $\hat{\kappa}_{(-j)}$ is set equal to the smallest of the $\hat{\kappa}_{(i)}$'s that could be computed for the sample considered..

Similiarly, in a jackknife procedure to estimate p(t), it is sometimes the case that either $\hat{\kappa}_{(-j)}$ does not exist or $\hat{p}_{(-j)}(t)$ exceeds 1 for the reduced data set that omits the j^{th} subgroup. In this case $\hat{p}_{(-j)}$ is set equal to the largest of those $\hat{p}_{(-i)}$'s that could be computed and were less than 1.

4.1 Results of a Simulation Experiment to Estimate κ

In this subsection results are given of a simulation experiment to estimate κ. For each replication, three different estimates of κ were computed. Estimate I, $\hat{\kappa}_I$, is computed by numerically solving (2.2) by search starting with the initial value $\hat{\kappa}_H$ of (2.3). Estimate II, $\hat{\kappa}_{II}$ is obtained by jackknifing $\hat{\kappa}$ using ten subgroups; the mean of the pseudo-values is the estimate. Estimate III is obtained by jackknifing $\ln(1+\hat{\kappa})$ using ten subgroups; the inverse transform of the mean of the pseudo-values $e^{\bar{y}} - 1$ is used as the estimate of κ. In the cases of Estimates II and III, if the estimate is negative, it is set equal to 0. For each estimate, the average bias

$$B = \frac{1}{500} \sum_{i=1}^{500} (\hat{\kappa}_i - \kappa)$$

and the relative mean square error

$$\text{Rel MSE} = \frac{1}{500} \sum_{i=1}^{500} \left[\frac{\hat{\kappa}_i - \kappa}{\kappa}\right]^2$$

are computed where

$$\kappa = \mu - 1$$

in the case of exponential service times with mean $\frac{1}{\mu} < 1$.

Results of the simulation appear in Table 1. All of the estimates have about the same relative mean square error. Jackknife estimates II and III have smaller bias than the straightforward estimate I. Jackknifing $\hat{\kappa}$ itself rather than $\ln(\hat{\kappa}+1)$ gives the smallest bias. As $\frac{1}{\mu}$ increases all the estimates have increased relative mean square error.

A simulation study was conducted to compare the performance of different confidence interval procedures. For each replication three 80% confidence intervals were constructed. Interval procedure I is a normal confidence interval which uses the straightforward estimate of κ, $\hat{\kappa}_I$, as the point estimate; the estimate of the variance is the data version of the asymptotic variance in the central limit theorem (2.14);

$$\hat{\sigma}^2_{CLT} = \hat{\kappa}^2 \hat{\beta}^{-2} \frac{1}{n-1} \sum_i (e^{\hat{\kappa} x_i} - \hat{\phi}(\hat{\kappa}))^2 \tag{4.1}$$

where

$$\hat{\phi}(\hat{\kappa}) = \frac{1}{n} \sum_i e^{\hat{\kappa} x_i} \tag{4.2}$$

$$\hat{\beta} = \frac{1}{n} \sum_i e^{\hat{\kappa} x_i} - 1 - \kappa x_i e^{\hat{\kappa} x_i} \tag{4.3}$$

with $\hat{\kappa} = \hat{\kappa}_I$ in this case. The 80%-point of the normal distribution is used to construct the interval. Limits that are negative are set equal to 0.

Confidence interval procedure II is a jackknife confidence interval which jackknifes $\hat{\kappa}$ and uses the 80% point of the student t-distribution with 9 degrees of freedom. Limits that are negative are set equal to 0.

Confidence interval procedure III is a jackknife confidence interval which jackknifes $\ln(\hat{\kappa}+1)$ and uses the 80% point of the student t-distribution with 9 degrees of freedom to give a confidence interval for $\ln(\kappa+1)$. The inverse transformation of the endpoints of the interval gives an interval for κ; limits that are negative are set equal to 0.

Results of the confidence interval simulation appear in Table 2. Reported are the number of 500 intervals that cover the true value of κ, (C); the number of the 500 intervals such that the entire interval lies below κ, (B); and the number of the 500 intervals such that the entire interval lies above the true value, (H). The average length of the confidence intervals is also given.

The number of intervals that cover the true value of κ for procedure III is within $.80 \pm (1.96)\sqrt{\frac{1}{500}(.2)(.8)} = [.765, .835]$. All but one case of the normal confidence intervals of procedure I are outside this range. All but one case using confidence interval procedure II are inside this range. The average width of confidence intervals for procedures II and III are about the same. Thus, although the jackknife estimate $\hat{\kappa}_{III}$ is a little more biased than $\hat{\kappa}_{II}$, the coverage of the jackknife confidence interval for $\hat{\kappa}_{III}$ appears to be somewhat better than for jackknife confidence interval procedure II.

TABLE 1

Bias and Mean Relative Square Error for Estimates of κ

Distribution / Estimate	Exponential $\frac{1}{\mu} = 0.6$ $\kappa = .6667$		Exponential $\frac{1}{\mu} = 0.7$ $\kappa = .4286$		Exponential $\frac{1}{\mu} = 0.8$ $\kappa = .2500$		Exponential $\frac{1}{\mu} = 0.9$ $\kappa = .1111$	
	B (S.E) $(\frac{B}{\kappa})$	Rel MSE (S.E.)	B (S.E) $(\frac{B}{\kappa})$	Rel MSE (S.E.)	B (S.E) $(\frac{B}{\kappa})$	Rel MSE (S.E.)	B (S.E) $(\frac{B}{\kappa})$	Rel MSE (S.E.)
I	.1147 (.014) [.172]	.2350 (.021)	.0557 (.011) [.130]	.3292 (.029)	.0057 (.008) [.023]	.6026 (.054)	.0865 (.0067) [.779]	2.400 (.2134)
II	.0219 (.014) [.033]	.2256 (.019)	.0067 (.011) [.016]	.3095 (.025)	.0102 (.008) [.041]	.5382 (.046)	.0469 (.0063) [.422]	1.785 (.161)
III	.0533 (.014) [.080]	.2212 (.019)	.0160 (.011) [.037]	.3073 (.027)	.0268 (.008) [.1072]	.5576 (.049)	.0608 (.008) [.5472]	1.997 (.177)

Table 2

Confidence Intervals for κ

Procedure / Distribution	Coverage I B %	Coverage I (C) (%)	Coverage I H %	Coverage II B %	Coverage II (C) (%)	Coverage II H %	Coverage III B %	Coverage III (C) (%)	Coverage III H %	Width I AV (S.E.)	Width II AV (S.E.)	Width II AV (S.
Exponential												
$\frac{1}{\mu} = 0.6$	45	(363)	92	38	(408)	54	29	(395)	76	.6402	.8089	.77
$\kappa = .6667$	9	(72.6)	18.4	7.6	(81.6)	10.8	5.8	(79)	15.2	(.005)	(.0124)	(.01
Exponential												
$\frac{1}{\mu} = 0.7$	49	(380)	71	50	(409)	41	42	(399)	59	.5279	.6255	.61
$\kappa = .4286$	9.8	(76)	14.2	10	(81.8)	8.2	8.4	(79.8)	11.8	(.004)	(.009)	(.00
Exponential												
$\frac{1}{\mu} = 0.8$	28	(407)	65	38	(420)	42	33	(413)	54	.4441	.4831	.49
$\kappa = .2500$	5.6	(81.4)	13	7.6	(84)	8.4	6.6	(82.6)	10.8	(.005)	(.008)	(.00
Exponential												
$\frac{1}{\mu} = 0.9$	1	(429)	70	54	(408)	38	52	(392)	56	.3733	.3763	.39
$\kappa = .1111$	.002	(85.8)	14	10.8	(81.6)	7.6	10.4	(78.4)	11.2	(.005)	(.009)	(.0

4.2 Simulation Results for Estimating p(t)

In this subsection, results are given of a simulation experiment to estimate p(t). For each replication three estimates of p(t) are computed. Estimate I, $\hat{p}_I(t)$ is computed for each replication using formulas (2.2), (3.5) - (3.9). If $\hat{p}_I(t)$ exceeds 1, it is set equal to 1.

There are (at least two) possible ways to implement a jackknife procedure to estimate $p(t)$. Estimate II is obtained by jackknifing $\ln(\kappa+1)$; an estimate of κ is obtained by the inverse transformation of the mean of the pseudo-values $\bar{y}_{LK}$,

$$\hat{\kappa}_{II} = e^{\bar{y}_{LK}} - 1;$$

$\hat{\kappa}_{II}$ is used in formulas (3.5) - (3.9) to obtain the estimate $\hat{p}_{II}(t)$. If $\hat{p}_{II}(t)$ exceeds 1, it is set equal to 1.

Estimate III is obtained by jackknifing $\ln[t - \ln \hat{p}(t)]$; if $\hat{p}(t)$ exceeds 1 for a reduced data set that omits the j^{th} subgroup, the estimate of $\ln[t - \ln \hat{p}(t)]$ for that reduced data set is put equal to the smallest estimate that could be computed from the other reduced data sets. An estimate of $p(t)$ is obtained by inverse transformation of the average of the pseudo-values $\bar{y}_{LPR}$;

$$\hat{p}_{III}(t) = e^{t} e^{-\bar{y}_{LPR}}.$$

If $\hat{p}_{III}(t) > 1$, it is replaced by 1.

For each estimate, the average bias

$$B = \frac{1}{500} \sum_{i=1}^{500} [\hat{p}_i(t) - p(t)] \tag{4.4}$$

and the relative mean square error

$$\text{Rel MSE} = \frac{1}{500} \sum_{i=1}^{500} \left[\frac{\hat{p}_i(t) - p(t)}{p(t)} \right]^2 \tag{4.5}$$

are computed where

$$p(t) = \frac{1}{\mu} e^{-(\mu - 1)t}. \tag{4.6}$$

The results appear in Table 3.

TABLE 3

Mean Bias and Mean Relative Square Errors

for Estimates of p(t).

Distribution	Exponential $\frac{1}{\mu}=.6$, T=1 $p(1) = .3081$		Exponential $\frac{1}{\mu}=.7$, T=2 $p(2) = .2971$		Exponential $\frac{1}{\mu}=.8$, T=3 $p(3) = .3779$	
Estimate	B (S.E)	Rel MSE (S.E.)	B (S.E)	Rel MSE (S.E.)	B (S.E)	Rel MSE (S.E.)
I	.004 (.005)	.138 (.009)	.024 (.008)	.328 (.026)	.005 (.010)	.326 (.020)
II	.044 (.006)	.216 (.020)	.060 (.008)	.409 (.032)	.054 (.011)	.436 (.031)
III	.0138 (.005)	.137 (.009)	.040 (.008)	.354 (.030)	.034 (.011)	.397 (.029)

The entries for estimate II in Table 3 suggest that jackknifing κ and then computing the estimate of the probability using the jackknifed estimate of κ increases the bias and relative mean square error of the estimate over $p_I(t)$ the straight-forward estimate. The entries for estimate III suggest the jackknifing $\ln[t - \ln \hat{p}(t)]$ gives about the same relative mean square error as the original estimate $\hat{p}_I(t)$ but increases the bias somewhat.

A simulation experiment was conducted to investigate the performance of confidence intervals obtained by jackknifing $\ln[t - \ln \hat{p}(t)]$. For each simulation replication, the average and variance of the pseudo-values are computed;

$$\bar{y}_J = \frac{1}{10} \sum_{j=1}^{10} y_{(j)}$$

$$\sigma_J^2 = \frac{1}{9} \sum_{j=1}^{10} (y_{(j)} - \bar{y})^2 .$$

An 80% confidence interval for $\ln[t - \ln p(t)]$ is

$$[LPL, LPV] = \bar{y}_J \pm (1.383) \frac{1}{\sqrt{10}} \sqrt{\sigma_J^2}$$

where 1.383 is the 80% point for a Student's t-distribution with 9 degrees of freedom. The limits of the interval are inversely transformed to give a confidence interval for $p(t)$.

$$PL = \exp\{t - e^{LPV}\}$$

and

$$PV = \exp\{t - e^{LPL}\}.$$

If a confidence limit exceeds 1, the limit is set equal to 1.

Results of the confidence interval simulation appear in Table 4. Reported are the number of the 500 intervals that cover the true value p(t) (C); the number of the 500 intervals for which the upper limit PU is below p(t) (B); and the number of the 500 intervals such that the lower limit PL is above the true value (H). The average width of the confidence interval is also given.

Table 4.
80% Confidence Intervals for p(t)

	Coverage			Average Width
	B (%)	C (%)	H (%)	(standard error)
Exponential $\frac{1}{\mu}=.6$, T=1 p(1) = .3081	56 (11.2)	415 (83)	29 (5.8)	.325 (.006)
Exponential $\frac{1}{\mu}=.7$, T=2 p(2) = .2971	52 (10.4)	408 (81.6)	40 (8)	.477 (.009)
Exponential $\frac{1}{\mu}=.8$, T=3 p(3) = .3779	54 (10.8)	412 (82.4)	34 (6.8)	.597 (.010)

The coverage of the confidence intervals is within

$.80 \pm (1.96)\sqrt{\frac{(.8)(.2)}{500}} = .80 \pm .035 = [.765, .835]$. The width of the interval increases as $\frac{1}{\mu}$ increases.

4.3 Simulation Results for Estimating p(t) for Mixed Exponential Service Times

In this subsection, results are given of a simulation experiment to estimate $p(t) = P\{W>t\}$ in the case of mixed exponential service times. For each replication 50 random numbers are generated from a mixed exponential distribution with

$$P\{S>t\} = \frac{1}{2}e^{-\mu_1 t} + \frac{1}{2}e^{-\mu_2 t} \qquad , \qquad t>0.$$

The estimate $\ln[t - \ln\hat{p}(t)]$ is jackknifed with ten subgroups as before where $\hat{p}(t)$ is given by (3.5) and 80% confidence intervals are constructed. Each simulation has 500 replications.

The asymptotic distribution of the virtual waiting time, W, can be found by inverting the moment generating function (1.1); it is a mixture of two exponential random variables.

Table 5 reports results of three simulations.

Case I: $\quad \frac{1}{\mu_1} = .30 \quad \frac{1}{\mu_2} = .90, \quad T = 1,$

$E(S) = 0.6, \quad P\{W>T\} = .3404;$

Case II: $\quad \frac{1}{\mu_1} = .35 \quad \frac{1}{\mu_2} = 1.05, \quad T = 2,$

$E(S) = 0.7, \quad P\{W>T\} = .3459;$

Case III: $\quad \frac{1}{\mu_1} = .40 \quad \frac{1}{\mu_2} = 1.20, \quad T = 3,$

$E(S) = 0.8, \quad P\{W>T\} = .4333;$

For each simulation, the mean bias and relative mean square error (4.4) and (4.5) are computed with $p(t) = P\{W>t\}$. Confidence interval coverage and average widths are also computed.

Both cases II and III each had one replication for which using all the data to compute the estimate $\hat{p}(t)$ of (3.5) resulted in a value larger than 1; these two replications are not counted in the summary statistics in Table 5.

Comparison of the mean biases and mean relative square errors of Table 5 with those of estimate III in Table 3 shows that they are about the same for both service time distributions. The coverage of the confidence intervals in Table 5 is again within [.765, .835]. However, the average length of the confidence intervals for the mixed exponential cases are larger than those reported in Table 4 for the exponential distributions.

Table 5

Results for Estimates of P{W>t}
for an M/G/1 Queue with Mixed
Exponential Service Times

Distribution Case	Estimate Bias (S.E)	Estimate Rel MSE (S.E.)	Confidence Intervals Coverage B %	(C) (%)	H %	Width Average (S.E.)
I	.024	.172	58	(407)	35	.392
	(.006)	(.013)	11.6	(81.4)	7	(.007)
II	.040	.384	63	(408)	28	.500
	(.009)	(.028)	12.6	(81.4)	5.6	(.010)
III	.022	.350	54	(411)	34	.637
	(.011)	(.021)	10.8	(82.4)	6.8	(.011)

4.4 Simulation Results for Estimating P{W>t} for Gamma Distributed Service Times

In this subsection, results are given of a simulation experiment to estimate p(t) = P{W>t} in the case of gamma service times. Each experiment has 500 replications. For each replication, 50 service times are generated having the distribution of the sum of two exponential random variables each having mean $\frac{1}{\mu}$. The estimate $\ln[t - \ln \hat{p}(t)]$ is jackknifed with ten subgroups as before where $\hat{p}(t)$ is given by (3.5) and 80% confidence intervals are constructed.

The asymptotic distribution of the virtual waiting time, W, can be found by inverting the moment generating function (1.1); it is a mixture of two exponential random variables.

Table 6 reports results of three simulations.

Case I: $\frac{1}{\mu} = .30$ $T = 1$,

$E(S) = 0.6$, $P\{W>T\} = .2508$;

Case II: $\frac{1}{\mu} = .35$ $T = 2$,

$E(S) = 0.7$, $P\{W>T\} = .2232$;

Case III: $\frac{1}{\mu} = .40$ $T = 3$,

$E(S) = 0.8$, $P\{W>T\} = .2950$;

Table 6

Results for Estimates of P{W>t}
for an M/G/1 Queue with Gamma
Service Times

Distribution Case	Estimate Bias (S.E)	Estimate Rel MSE (S.E.)	Confidence Intervals Coverage B %	Coverage (C) (%)	Coverage H %	Width Average (S.E.)
I	.007	.110	61	(401)	38	.227
	(.004)	(.007)	12.2	(80.2)	7.6	(.003)
II	.0167	.299	61	(397)	42	.328
	(.005)	(.026)	12.2	(79.4)	8.4	(.007)
III	.0251	.390	62	(407)	31	.490
	(.008)	(.034)	12.4	(81.4)	6.2	(.010)

The mean bias and relative mean square error are about the same as for the exponential and mixed exponential service time distributions. The confidence interval coverage is once again within [.765. .835]. The average width of the confidence intervals is smaller than the widths in the exponential and mixed exponential cases. This is to be expected since the gamma distribution has a shorter tail than the exponential or mixed exponential distribution.

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